TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 181, July 1973

IMPROBABILITY OF COLLISIONS IN NEWTONIAN GRAVITATIONAL SYSTEMS.II

BY

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ABSTRACT. It is shown that the set of initial conditions leading to collision in the inverse square force law has measure zero. For the inverse q force law the behavior of binary collisions for $1 < q < 3$ and the behavior of any collision for $q = 1$ is developed. This information is used to show that collisions are improbable in the inverse q force law where $q < 17/7$ and that binary collisions are improbable for $q < 3$.

How abundant are collisions in the n-body problem? It has been known for some time that the set of initial conditions leading to a single binary collision has measure zero. J. E. Littlewood [1, Problem 13] raised the question whether *all* collisions of point masses are improbable in this measure theoretic sense. Intuitively this would seem to be so as it seems there is a binary collision "contained" in a larger collision. As Littlewood implied, it would make an interesting paradox if it were false.

In an earlier paper [4] it was announced that the set of initial conditions leading to collisions has measure zero. However, due to an error [5], the analysis holds for all cases except multiple binary collisions, triple collisions, or a simultaneous triple and binary collision. It is one of the purposes of this communication to settle these remaining cases. The ideas are essentially the same as in [4] but sharper estimates are needed to handle these more delicate cases.

A natural question is whether this result is peculiar to the inverse square force law. In §§4–8, this question will be studied in more detail. It will be shown that the result holds in a wide class of inverse q force laws. The most difficult part of the analysis is to determine the analytical behavior of collisions.

The basic assumptions are that n is finite and that the motion is in an inertial coordinate system. Let m_i, $\mathbf{r}_i$, $\mathbf{v}_i$ be respectively the mass, position vector and velocity vector of the ith particle. The same letter will be used to denote the magnitude of a vector, e.g. $r_i = |\mathbf{r}_i|$, $r_{ik} = |\mathbf{r}_i - \mathbf{r}_k|$. By assuming that the gravitational constant is equal to unity the equations of motion become

Received by the editors December 1, 1971 and, in revised form, August 25, 1972.

AMS (MOS) subject classifications (1970). Primary 70F10, 70F35, 34C35; Secondary 28A65, 58F05, 70F15, 70F05, 26A84.

Key words and phrases. n-body problem, gravitational systems, collision, measure preserving, singularities, celestial mechanics.

(1) Supported by National Science Foundation grant GP32116.

$$m_i\ddot{\mathbf{r}}_i = \sum m_i m_k(\mathbf{r}_k - \mathbf{r}_i)/r_{ik}^3. \tag{1.1}$$

We say that there is a *collision at time* $t = t_0$ *if* $\mathbf{r}_i \to \mathbf{L}_i$ *as* $t \to t_0$ *where at least two of the limits are the same.*

It is known that particles colliding at $t = t_0$ approach each other like $|t - t_0|^{2/3}$ [2]. However, if there are only two or three particles colliding at a given point, then the literature gives an improved estimate which will be used here. Assume without loss of generality that $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_k$ collide with a common limit as $t \to t_0$ and any other colliding particles have different limit points. If $k = 2, 3$, it is known that as $t \to t_0$, $\mathbf{r}_i - \mathbf{L}_i = \mathbf{c}_i|t - t_0|^{2/3} + o(t - t_0)$ and

$$\mathbf{v}_i = (2/3)\mathbf{c}_i|t - t_0|^{-1/3} + O(1) \tag{1.2}$$

where

$$-\frac{2}{9}\mathbf{c}_i = \sum_{j=1;j\neq i}^{k} \frac{m_j(\mathbf{c}_j - \mathbf{c}_i)}{c_{ij}^3}, \qquad i = 1, 2, \cdots, k. \tag{1.3}$$

For $k = 2$ this can be found in [8] or §7. For $k = 3$ the statement follows from a modification of [6]. (However, this will not be needed here.)

We first discuss solutions of the equations (1.3). The solution defines a configuration in R^3. We say that two solutions are the same if they are congruent in the sense of Euclidean geometry. With this identification it is known that (see [9, pp. 273–279] for example) for $k = 3$ there are 4 solutions: an equilateral triangle and three collinear solutions which depend on the arrangement of the masses. In all cases the center of mass is located at the origin, i.e. $\Sigma^k m_i\mathbf{c}_i = 0$.

In the two body case there is only one solution, $\mathbf{c}_1 = -A\mathbf{c}_2$ where A is a positive constant in terms of the masses.

(While similar results are suspected for $k \geq 4$ they have not been shown. It is known that the $\mathbf{r}_i$'s tend to configurations defined by (1.3), but how fast is *not* known. In fact for arbitrary k and m_i, it is not known even whether there are a finite number of solutions of (1.3)!)

The idea of the proof is basically quite simple. The system is measure preserving so the measure of the set of initial conditions leading to a collision at time t_0 will have an upper bound determined by how fast $|\mathbf{r} - \mathbf{L}_i| \to 0$ and $\mathbf{v}_i \to \infty$. It turns out that this upper bound can be made arbitrarily small, leading to the stated result.

2. Statement of theorem.

Theorem 1. *The set of initial conditions leading to a simultaneous multiple binary collision, a triple collision, or a simultaneous triple and binary collision has measure zero.*

In combination with [4] this now yields the announced result:

Theorem 2. *The set of initial conditions leading to collisions has measure zero.*

Actually, as will be seen in the proof, the statement can be made much stronger. Theorem 2 also holds for the planar n-body problem. (The ideas in the proof are the same, the estimates change only by the dimension. Only the case of a simple binary collision requires more care. Here we need the ideas and estimates found in §7.) Theorem 2 does not hold for the linear n-body problem as it can be easily shown that in this setting *all* initial conditions lead to a collision. However, it can be shown by use of techniques employed here that in the linear n-body problem, it is improbable that a k body collision will occur where $k > 4$.

Finally, all the theorems and corollaries in this paper are stated as results for the full phase space with none of the integrals fixed. However, as will be seen from the proof contained in §3, a simple modification in the proofs obtains the same results for a fixed center of mass (and the lower dimensional measure in the "reduced" phase space). However, for $n \geq 3$, it is not known, nor has it been investigated by the author, whether the statement holds for a fixed energy or fixed angular momentum surface. The statement need not be the same. For example, in the two body problem, *all* initial conditions with zero angular momentum lead to a collision.

As was seen in [4], the more particles participating in a collision, the more leeway is available in the derived estimates. It turns out to be the same here. The binary collision estimates are more delicate than those needed in the proof for triple collisions, which in turn involves more care than the estimates used in a simultaneous triple and binary collision.

In this section it will be shown that the set of initial conditions leading to simultaneous binary collisions has measure zero. The case of a simultaneous binary collision and a multiple collision follows immediately from this proof. The case of higher order collisions will be considered in the next section.

As in [4], the problems are that: (1) the collision may occur anywhere in R^3, (2) the collision may occur at any time, and (3) the location of the noncolliding particles may be anywhere in space. We overcome these difficulties by dividing the appropriate spaces into a countable number of bounded sets.

Proof of simultaneous binary collisions. We will solve the problem for two simultaneous binary collisions. The cases of a single binary and multiple collisions or multiple binary collisions follow immediately from this proof. The case of a single binary will be proved in a more general setting in §7. Assume that $\mathbf{r}_1$ and $\mathbf{r}_2$ collide at the same time that $\mathbf{r}_3$ and $\mathbf{r}_4$ collide at a distinct limit point. Let R_1^6 be the subspace $(\mathbf{r}_1, \mathbf{r}_2, 0, \cdots, 0)$, R_2^6 be the subspace $(0, 0, \cdots, 0, \mathbf{v}_1, \mathbf{v}_2, 0, \cdots, 0)$, R_3^6 be the subspace $(0, 0, \mathbf{r}_1, \mathbf{r}_2, 0, \cdots, 0)$, and R_4^6 the subspace

$(0, 0, \cdots, v_3, v_4, 0, \cdots, 0)$. Let $R^{6(n-4)}$ be the orthogonal complement of $R_1^6 \times R_2^6 \times R_3^6 \times R_4^6$.

Let L_1 and L_2 be arbitrary unit cubes on $\mathbf{r}_1 = \mathbf{r}_2$ in R_1^6 and on $\mathbf{r}_3 = \mathbf{r}_4$ in R_3^6 respectively. This will be where the respective binary collisions occur.

Let p be an arbitrary point in $R^{6(n-4)}$. The $6(n-4)$ dimensional sphere of radius one about p, $S(p, 1)$, will fix the positions and velocities of the noncolliding particles.

Notice that, for each positive integer α, $(2/3)(2^{\alpha/3})(0, \cdots, \mathbf{c}_1, \mathbf{c}_2, 0, \cdots 0)$ defines a smooth two dimensional manifold in R_2^6. We denote this manifold as $\mathfrak{M}_2(\alpha)$. (The $\mathbf{c}_i$'s were introduced in (1.3).) This is so as the magnitudes of $\mathbf{c}_1$ and $\mathbf{c}_2$ are fixed. Once $\mathbf{c}_1$ is given, $\mathbf{c}_2$ is uniquely determined. ($\mathbf{c}_2 = -B\mathbf{c}_1$, where B is a positive constant.) Everything depends upon the three space orientation of the vector $\mathbf{c}_1$. A similar statement applies to R_4^6 and its manifold $\mathfrak{M}_4(\alpha)$ which is defined by $\mathbf{c}_3$ and $\mathbf{c}_4$.

For arbitrary time $t \neq 0$ and positive integer α, define $B^{\alpha}(t)$ to be the set of points in R^{6n} such that the following hold.

1. Viewing a point in $B^{\alpha}(t)$ as an initial condition of (1.1) at time t, the solution exists in the time interval $[0, t]$ (or $[t, 0]$).

2. The components which are in $R^{6(n-4)}$ lie in $S(p, 1)$, i.e. in the sphere of radius one about p.

3. The components in R_1^6 are within distance $4(|\mathbf{c}_1| + |\mathbf{c}_2|)(2^{-\alpha})^{2/3}$ of L_1 and those in R_3^6 are within distance $4(|\mathbf{c}_3| + |\mathbf{c}_4|)(2^{-\alpha})^{2/3}$ of L_2.

4. The components in R_j^6 are within distance $(2^{-\alpha})^{-1/60}$ of $\mathfrak{M}_j(\alpha)$ where $j = 2, 4$.

Next it will be shown that $B^{\alpha}(t)$ is a measurable set. To see this, note that conditions 2, 3, and 4 clearly define a measurable set. That condition 1 defines a measurable set follows from the facts that (i) some $\mathbf{v}_i$ becomes unbounded at singularities of (1.1) and (ii) the solutions of (1.1) are continuous with respect to initial conditions. Statement (ii) implies that if $q \in B^{\alpha}(t)$, then there exists $\delta > 0$ such that if an initial condition is within δ distance of q, then its solution will differ from the one corresponding to q by at most 1 for all time that it exists in the time interval $[0, t]$. Since the solution with value q at time t exists in the interval $[0, t]$, its trajectory is bounded. Hence any initial condition at time t within δ distance of q has a bounded trajectory, which, in turn, implies that the solution exists in the time interval $[0, t]$. That is, (1) defines an open set. It follows that $B^{\alpha}(t)$ is measurable.

An upper bound on the measure of $B^{\alpha}(t)$ is easily computed. Condition 2 gives the $6(n-4)$ dimensional volume of $S(p, 1)$. The six dimensional volume of the set in R_j^6, $j = 1, 3$, defined by 3 is bounded above by

$$|1 + 4(|\mathbf{c}_j| + |\mathbf{c}_{j+1}|)(2^{-\alpha})^{2/3}|^3(4(|\mathbf{c}_j| + |\mathbf{c}_{j+1}|)(2^{-\alpha})^{2/3})^3 \le D_j(2^{-\alpha})^2$$

where D_j is some positive constant independent of α.

The six dimensional volume of the set in R_j^6, $j = 2, 4$, defined by condition 4, is bounded above by $D_j(2^{-\alpha})^{-2/3}(2^{-\alpha})^{-4/60} = D_j(2^{-\alpha})^{-11/15}$. Again, D_j is some positive constant independent of α.

(2.1) Hence, $m(B^{\alpha}(t)) \le E(2^{-\alpha})^{38/15}$, where E is some positive constant independent of α.

We now choose an arbitrary unit interval of time, say $[1, 2]$, and divide it into 2^{β} equal parts where β is the first integer greater than or equal to $4\alpha/3$. The partition points are then labeled $t_0 = 1, \cdots, t_{2^\beta} = 2$.

Define $B^{\alpha}[1, 2]$ to be the set of initial conditions (at time 0) whose solutions are eventually in $\bigcup_{i=0}^{2^\beta - 1} B^{\alpha}(t_i)$. We claim that $B^{\alpha}[1, 2]$ is measurable. Since the Newtonian n-body problem is a conservative dynamical system, the system is measure preserving. Hence, the set of initial conditions leading to $B^{\alpha}(t_i)$ is measurable and has the same measure as $B^{\alpha}(t_i)$. But $B^{\alpha}[1, 2]$ is the finite union of these measurable sets. This implies that it is measurable and

$$m(B^{\alpha}[1, 2]) \le \sum_{i=0}^{2^\beta - 1} m(B^{\alpha}(t_i)) = 2^{\beta}E(2^{-\alpha})^{38/15} \tag{2.2}$$

$$\le 2^{4(\alpha+1)/3}E(2^{-\alpha})^{38/15} = 2E(2^{-\alpha})^{18/15}.$$

Define $B[1, 2] = \limsup B^{\alpha}[1, 2]$ as $\alpha \to \infty$. $B[1, 2]$ is a measurable set. By definition, for any N, $B[1, 2] \subset \bigcup_{\alpha=N}^{\infty} B^{\alpha}[1, 2]$. Thus

$$m(B[1, 2]) \le \sum_{\alpha=N}^{\infty} m(B^{\alpha}[1, 2]) \le 2E \sum_{\alpha=N}^{\infty} (2^{-38/15})^{\alpha}.$$

Since the right-hand side is a convergent series, it follows that $m(B[1, 2]) = 0$.

We now show how set $B[1, 2]$ relates to the collision problem. Let $\mathbf{C}$ be the set of initial conditions which:

(1) suffer their first singularity at time $t^* \in [5/4, 7/4]$,

(2) the singularity is due to simultaneous binary collisions between particles $\mathbf{r}_1, \mathbf{r}_2$ and particles $\mathbf{r}_3, \mathbf{r}_4$,

(3) the common limit point of $\mathbf{r}_1$ and $\mathbf{r}_2$ is in L_1, the common limit point of $\mathbf{r}_3$ and $\mathbf{r}_4$ is in L_2,

(4) the noncolliding particles have their limit in $S(p, 1/2)$.

We first justify condition 4 of the above definition by showing that the velocity of a noncolliding particle approaches a definite limit as $t \to t^*$. By (1.1), $|\ddot{\mathbf{r}}| \le \Sigma\, m_j r_{ij}^{-2}$. If particle i does not participate in a collision at $t = t^*$, then for each j, r_{ij} is bounded away from zero in the time interval $[0, t^*)$. Hence $|\ddot{\mathbf{r}}| = O(1)$ as $t \to t^*$. By integrating both sides from t_1 to t_2, $t_1 < t_2 < t^*$, it follows that $|\dot{\mathbf{r}}_i(t_2) - \dot{\mathbf{r}}_i(t_1)| = O(|t_2 - t_1|)$. As $t_1, t_2 \to t^*$, the right-hand side goes to zero, forcing the left-hand side to do likewise. By the Cauchy criterion for the existence of a limit, $\dot{\mathbf{r}}_i \to \mathbf{A}_i$ as $t \to t^*$ for $i = 5, 6, \cdots, n$. Likewise, $\mathbf{r}_i \to \mathbf{L}_i$ as $t \to t^*$ for $i = 5, 6, \cdots, n$.

Claim. $\mathbf{C} \subset B[1, 2]$.

To prove this, we show that if $q \in \mathbf{C}$, then $q \in B^{\alpha}[1, 2]$ for an infinite number of choices of α.

If $q \in \mathbf{C}$, then (by condition 4) it defines a vector $(\mathbf{0}, \cdots, \mathbf{0}, \mathbf{L}_5, \cdots, \mathbf{L}_n, \mathbf{0}, \cdots, \mathbf{0}, \mathbf{A}_5, \cdots, \mathbf{A}_n) \in S(p, 1/2)$. This implies that after some time $(\mathbf{0}, \cdots, \mathbf{0}, \mathbf{r}_5, \cdots, \mathbf{r}_n, \mathbf{0}, \cdots, \mathbf{0}, \mathbf{v}_5, \cdots, \mathbf{v}_n)$ is in $S(p, 1)$. Hence for all partition points between this time and t^*, condition 2 of $B^{\alpha}(t_i)$ is satisfied. But for all α greater than some integer, such partition points can always be found.

Colliding particles $\mathbf{r}_1$ and $\mathbf{r}_2$ approach their common limit point $\mathbf{L}_1 \in L_1$ in such a fashion that $|\mathbf{r}_i - \mathbf{L}_1| = |\mathbf{c}_i|(1 + o(1))|t - t^*|^{2/3}$. Hence, after some time

$$|\mathbf{r}_i - \mathbf{L}_1| \leq (|\mathbf{c}_1| + |\mathbf{c}_2|)|t - t^*|^{2/3}. \tag{2.3}$$

However, for all α greater than some integer, $8(2^{-\alpha})$ is smaller than the magnitude of the difference between t^* and this time. For all partition points t_i, such that $0 < t^* - t_i < 8(2^{-\alpha})$, it follows that $|t^* - t_i|^{2/3} < 4(2^{-\alpha})^{2/3}$. From (2.3), it follows that for all α greater than some integer, there exist partition points which satisfy condition 3 in the definition of $B^{\alpha}(t_i)$.

A similar statement holds for particles $\mathbf{r}_3$ and $\mathbf{r}_4$.

Notice that if α is large enough, the above two conditions are satisfied for *any* partition point satisfying $0 < t^* - t_i < 8(2^{-\alpha})$. To satisfy the remaining condition in the definition of $B^{\alpha}(t_i)$ we select a unique partition point from this set.

Since $\mathbf{v}_i = (2/3)\mathbf{c}_i|t - t^*|^{-1/3} + o(1)$, after some time $||\mathbf{v}_i| - (2/3)|\mathbf{c}_i|\,|t - t^*|^{-1/3}| < 1$. That is, for all α greater than some integer, $0 < |t - t^*| \leq 8(2^{-\alpha})$ implies the above. For each α greater than this integer choose the unique partition point t_m such that $2^{-\alpha} \leq t^* - t_m < 2^{-\alpha} + 2^{-\beta}$. Then

$$\begin{aligned}
&||\mathbf{v}_i(t_m)| - (2/3)|\mathbf{c}_i|(2^{-\alpha})^{-1/3}| \\
&\qquad \leq ||\mathbf{v}_i(t_m)| - (2/3)|\mathbf{c}_i||t^* - t_m|^{-1/3}| + (2/3)|\mathbf{c}_i||(2^{-\alpha})^{-1/3} - (t^* - t_m)^{-1/3}| \\
&\qquad \leq 1 + (2/3)|\mathbf{c}_i||(2^{-\alpha})^{-1/3} - (2^{-\alpha} + 2^{-\beta})^{-1/3}|.
\end{aligned}$$

For all α, and hence β, greater than some integer, the last term is bounded by $1 + 3|\mathbf{c}_i|(2^{-\alpha})^{-4/3}2^{-\beta}$. Since $\beta \geq 4\alpha/3$ it follows that the above is bounded by $1 + 3|\mathbf{c}_i|(2^{4\alpha/3})2^{-4\alpha/3} < 2^{\alpha/60}$ for α greater than some integer. This is condition 4 in the definition of $B^{\alpha}(t_m)$.

The above shows that for all α greater than some integer, $q \in \mathbf{C} \Rightarrow$ the solution corresponding to initial condition q is in $B^{\alpha}(t_m) \Rightarrow q \in B^{\alpha}[1, 2] \Rightarrow q \in B[1, 2]$

or that $\mathbf{C} \subset B[1, 2]$. This implies that $m\mathbf{C} = 0$.

The set $\mathbf{C}$ depends upon the location of unit cubes L_1 and L_2, the specified time interval during which the collisions can occur (of length $1/2$), and the limiting values of the noncolliding particles in $S(p, 1/2)$ (and hence on p in $R^{6(n-4)}$). To obtain all initial conditions leading to a simultaneous double binary collision we sum the $\mathbf{C}$'s over a countable number of unit cubes which cover $\mathbf{r}_1 = \mathbf{r}_2$ and $\mathbf{r}_3 = \mathbf{r}_4$, a countable number of time intervals of length $1/2$ which cover the real time interval $(-\infty, \infty)$ and over all rational $p \in R^{6(n-4)}$ to obtain set $\mathbf{D}$.

Clearly, $\mathbf{D}$ contains all initial conditions leading to a simultaneous collision of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, $\mathbf{r}_4$. Since $\mathbf{D}$ is the countable union of sets of measure zero, $m\mathbf{D} = 0$. Since n is finite, the permutation of the indices of the colliding particles is finite. Hence the set of initial conditions leading to simultaneous binary collisions has measure zero.

3. **Multiple collisions.** The estimates in the last section depended upon the sharp estimates concerning the behavior of the velocities in a binary collision. While this information can be obtained for triple collisions (at least to the accuracy demanded by this analysis) it is missing for multiple collisions involving 4 or more bodies.

To provide for this lack of information, a second approach will be used. This approach differs from that of §2 in that new coordinates are introduced. In this new coordinate system, the unit cube L can be eliminated. This has the advantage of adding additional terms which are going to zero. These additional terms cancel the effects of the crude estimates on the behavior of the velocity.

This approach has the further advantage of supporting the statement following Theorem 2. That is, these results hold even if the center of mass is fixed. In other words, if the center of mass is fixed, then we can talk about a reduced phase space of dimension $6n - 6$. In this reduced phase space, collisions are still improbable.

The coordinates employed are due to Jacobi and will be defined as follows. Let $\rho_1 = \mathbf{r}_2 - \mathbf{r}_1$. Let ρ_2 be the vector from the center of mass of $\mathbf{r}_1$ and $\mathbf{r}_2$ to $\mathbf{r}_3$. Continuing in this fashion, let ρ_k be the vector from the center of mass of particles $\mathbf{r}_1, \cdots, \mathbf{r}_k$ to the vector $\mathbf{r}_{k+1}$, $k = 1, 2, \cdots, n-1$. Since $\sum m_i \mathbf{r}_i = \mathbf{c}t + \mathbf{d}$, it is clear that the behavior of $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$, $i = 1, \cdots, n$, completely determines the values of $\mathbf{c}$, $\mathbf{d}$, ρ_k and $\dot{\rho}_k$ and vice versa.

Since we are using Jacobi coordinates, we are tacitly assuming that $\mathbf{c}$ and $\mathbf{d}$ are fixed and that our reduced phase space has dimension $6n - 6$.

We now show that a k-fold collision has ($6n - 6$ dimension) measure zero where $k \geq 3$. With only minor modifications in the coordinate system, the proof for simultaneous multiple collisions follows in the same fashion. These modifications, plus modifications which will allow us to include the inverse q force law, will be indicated at the end of this section.

The proof uses the fact that if there is a collision at $t = t^*$ in the n-body problem, then there exists a positive constant C such that after some time

$$(3.1) \qquad |\mathbf{r}_i - \mathbf{L}_i| \leq C|t - t^*|^{2/3} \quad \text{and} \quad |\mathbf{v}_i| \leq C|t - t^*|^{-1/3}.$$

(See [2], [4], or [3, p. 239].) Assume that particles $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_k$ suffer a k-fold collision at t^*. That is, they all share the same limit point as $t \longrightarrow t^*$ and the remaining particles have distinct limits. Then it follows from the definition of the Jacobi coordinates that there exists a positive constant D, which depends only on the masses and C, such that

$$(3.2) \qquad |\boldsymbol{\rho}_i| \leq D|t - t^*|^{2/3} \quad \text{and} \quad |\dot{\boldsymbol{\rho}}_i| \leq D|t - t^*|^{-1/3}.$$

(Note that in addition to the difficulties listed prior to the proof in §2, in this setting there is the additional problem that the value of D *may* depend upon the initial conditions.)

We now begin the proof. Let R^{6n-6} be the phase space for the Jacobi coordinates. Define $R_1^{3(k-1)}$ to be the subspace with vectors $(\rho_1, \cdots, \rho_k, 0, \cdots, 0)$. Define $R_2^{3(k-1)}$ to be the subspace with zeros in all components except for $\dot{\rho}_1, \cdots, \dot{\rho}_{k-1}$. Let $R^{6(n-k)}$ be the orthogonal complement of the product of these two spaces. Let p be an arbitrary point in $R^{6(n-k)}$ and let B be an arbitrary positive integer.

For $t \neq 0$ and α a positive integer, define $B^{\alpha}(t)$ to be the set of points in R^{6n-6} such that:

(1) if the point is viewed as an initial condition at time t, the solution exists in the interval $[0, t]$ (or $[t, 0]$),

(2) the components in $R^{6(n-k)}$ are in $S(p, 1)$,

(3) the magnitude of the components in $R_1^{3(k-1)}$ are less than $B(2^{-\alpha})^{2/3}$, and

(4) the magnitude of the components in $R_2^{3(k-1)}$ are less than $B(2^{-\alpha})^{-1/3}$.

Using the same arguments as given in §2, it follows that $B^{\alpha}(t)$ is measurable and $mB^{\alpha}(t) \leq E(2^{-\alpha})^{k-1}$ where E is some positive constant indepentent of α.

We now divide an arbitrary unit interval, say $[1, 2]$ into $2^{\alpha+5}$ equal parts and define partition points $t_0 = 1, \cdots, t_{2^{\alpha+5}} = 2$. In the same fashion as before, we define sets $B^{\alpha}[1, 2]$ and $B[1, 2]$. The crucial stage in the proof that $mB[1, 2] = 0$ is to show that $mB^{\alpha}[1, 2] \leq F(2^{-\alpha})^g$ where g is some positive number. In this case $F = 2^5E$ and $g = k - 2$. Since $k \geq 3$, g is positive and $mB[1, 2] = 0$.

We define $\mathbf{C}$ to be the set of initial conditions such that

(1) the first singularity occurs at time $t^* \in [5/4, 7/4]$,

(2) the first singularity is a k-fold collision between particles $\mathbf{r}_1, \cdots, \mathbf{r}_k$,

(3) the value of D, as defined in (3.2), is less than $B/2$, and

(4) the limit of the noncolliding particles is in $S(p, 1/2)$.

The proof that $\mathbf{C} \subset B[1, 2]$, and hence that $m\mathbf{C} = 0$, is the same as in § 2 except for the velocity terms of the colliding particles. We find that if $q \in \mathbf{C}$, then for all α greater than some integer, the solution corresponding to initial condition q satisfies the first three conditions of the definition $B^{\alpha}(t_i)$ for all partition points satisfying $0 < t^* - t_i < (2^{-\alpha})$. Let t_m be the unique partition point satisfying $2^{-\alpha} - 2^{-(\alpha+5)} \le t^* - t_m \le 2^{-\alpha}$. If $\dot{\rho}_{ix}$ stands for a component of $\dot{\rho}_i$, then for all α larger than some integer

$$|\dot{\rho}_{ix}(t_m)| \le 2(B/3)(t_m - t^*)^{-1/3} \le (2/3)B(2^{-\alpha} - 2^{-(\alpha+5)})^{-1/3} \le B(2^{-\alpha})^{-1/3}.$$

Thus $q \in \mathbf{C}$ implies its solution is in $B^{\alpha}(t_m)$ for all α larger than some integer. This in turn shows that $m\mathbf{C} = 0$.

In this setting, set $\mathbf{C}$ depends upon the value of B, the time interval of length $1/2$ and the value of p. We sum over all rational p, positive integers B, and a countable number of time intervals of length $1/2$ which cover $(-\infty, \infty)$ to complete the proof.

In order for this proof to apply to the full phase space we simply use the constants $\mathbf{c}$ and $\mathbf{d}$ which define the center of mass and apply six times the well-known theorem of Fubini which asserts that a measurable set $\mathbf{D} \subset R^p = R^1 \times R^{p-1}$ must have measure zero if it intersects each hyperplane (constant) $\times R^{p-1}$ in a set of $(p-1)$-dimensional measure zero. This completes the proof.

The simplest way to handle simultaneous multiple collisions is to modify the Jacobi coordinate system. We describe the modifications for a simultaneous k-fold and p-fold collision. The modifications for a more complicated situation follow immediately. Let $\mathbf{r}_1, \cdots, \mathbf{r}_k$ share a limit point as $t \to t^*$ and $\mathbf{r}_{k+1}, \cdots, \mathbf{r}_{k+p}$ share a second limit while the remainder of the particles have distinct limits. Define ρ_i, $i = 1, 2, \cdots, k$, as before. Let $\rho_{k+i-1} = \mathbf{r}_{k+i} - \mathbf{r}_{k+1}$, $i = 2, \cdots, p$. Define ρ_i, $i = k + p, \cdots, n-1$, as before. With this coordinate system it is seen quite easily that the additional collision improves the estimates in the proof.

Finally, if there is a collision at $t = t^*$ in the inverse q force law where $1 < q < 3$, then there exists a positive constant D such that after some time

$$|\mathbf{r}_i - \mathbf{L}_i| \le D|t - t^*|^{2/(q+1)} \quad \text{and} \quad |\dot{\mathbf{r}}_i| \le D|t - t^*|^{(1-q)/(1+q)}.$$

(See [3, p. 239].) The above analysis will hold for a k-fold collision if $q < (9k - 10)/(3k - 2)$. Thus

Corollary 1. *In the inverse q force law, $1 < q < 3$, the set of initial conditions leading to a collision which includes (or consists of) a k-fold collision has measure zero if $k - 1 > (q + 1)/(9 - 3q)$.*

Notice that for large k, q may be chosen close to 3. Also note that for $k = 3$, q may be any value less than $17/7$.

4. **The inverse q force law.** Is this result concerning the improbability of collisions peculiar to the inverse square force law? It was shown in [4] that for the inverse q force law, where $q \geq 3$, there exist open sets of initial conditions which lead to collisions. Thus for the inverse q force law there is a bound on q for an improbability of collisions statement. It is the belief of the author that this bound is sharp and that collisions are improbable for the inverse q force law when $q < 3$.

As we have seen, the case of binary collisions is the most delicate to prove. The above conjecture is thus supported by:

Theorem 3. *In the inverse q force law where $q < 3$, the set of initial conditions leading to a binary collision has measure zero.*

Corollary 2. *In the inverse q force law, $q < 3$, the set of initial conditions leading to a simultaneous collision which contains at least one binary collision has measure zero.*

The proof of this theorem depends on the analytic behavior of binary collisions which will be developed in §7. The analytic behavior of higher order collisions is probably much the same as binaries. If so, the conjecture is true. However, there are difficulties in proving this statement which will be briefly discussed in §8.

For certain values of q, cruder statements about the analytic behavior of collisions will suffice to prove an improbability of collisions statement.

Theorem 4. *In the inverse q force law where $q < 17/7$ the set of initial conditions leading to collisions has measure zero.*

The proofs of these theorems will be given in the following order. For the remainder of this section Theorem 4 will be proved for $q < 1$. §§5 and 6 will handle the case $q = 1$. In §7, Theorem 3 and Corollary 2 will be proved for $1 < q < 3$, since the case $q \leq 1$ will have been shown. From Corollary 2, it follows that for $1 < q$, Theorem 4 need only be proved for collisions containing triple and higher order collisions. But from Corollary 1 it is known that these collisions are improbable for $1 < q < 17/7$.

For the inverse q force law the equations of motion are

$$m_j\ddot{\mathbf{r}}_i = \sum_{j=1;j\neq 1}^{n} \frac{m_i m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^{q+1}} = \frac{\partial U}{\partial \mathbf{r}_i}. \tag{4.1}$$

For $q \neq 1$,

$$U = \frac{1}{q-1} \sum{}^{*} \frac{m_i m_j}{r_{ij}^{q-1}} \tag{4.2}$$

and the summation $(*)$ is $1 \leq i < j \leq n$. The conservation of energy integral is then

$$T = \tfrac{1}{2} \sum m_i \mathbf{v}_i^2 = U + h. \tag{4.3}$$

Note that for $q < 1$, U is negative. Since T is nonnegative by definition, the constant of integration h is positive and $T \leq h$. Hence, for all time that the motion is defined, $|\dot{\mathbf{r}}_i| \leq (2h/m_i)^{1/2}$. Thus, if there is a collision at time t^*, $|\mathbf{r}_i - \mathbf{L}_i| \leq A|t - t^*|$ where A is a positive constant depending on the masses and h. This estimate combined with the fact that the velocities are bounded as a collision is approached is sharp enough so that the analysis used in §3 will yield the stated result.

(Note that a classification of motion as $t \to \infty$ in the sense of [3] is trivial for $q < 1$. Namely, *all motion is bounded for all time.* This is so as T is nonnegative. Hence from (4.3) we have $\Sigma^* m_i m_j r_{ij}^{1-q} \leq (1-q)h$. That is, the mutual distances r_{ij} are bounded. It follows immediately that the distance of $\mathbf{r}_i$ from the center of mass is bounded above for all time.)

5. **The inverse force law.** For $q = 1$ we will derive the analytic behavior of particles at collision. In this setting the definition of U in (4.1) and (4.3) is $U = -\Sigma^* m_i m_j \ln r_{ij}$.

Since the system is invariant with respect to time translates and is time reversible, we will assume without loss of generality that there is a collision at time $t = 0$ and that it is approached through positive values of time. Also, we assume $\Sigma m_i \mathbf{r}_i = \mathbf{0}$. As $t \to 0$, the particles approach limits, $\mathbf{L}_1, \mathbf{L}_2, \cdots, \mathbf{L}_k$ where $k < n$. Let $G_s = \{j | \ \mathbf{r}_j \to \mathbf{L}_s \text{ as } t \to 0\}$ and define $2J = \Sigma_{s=1}^{k} \Sigma_{j \in G_s} m_j (\mathbf{r}_j - \mathbf{L}_s)^2$. What will be shown is

Lemma 1. *As* $t \to 0$, $J = o(t^2 (\ln t^{-1})^{\beta})$ *and* $T = o((\ln t^{-1})^{1+\epsilon})$ *where* β *is any constant greater than one and* $\epsilon > 0$.

From the lemma and the definitions of J and T it follows that $|\mathbf{r}_i - \mathbf{L}_s| = o(t(\ln t^{-1})^{\beta/2})$ and $|\mathbf{v}_i| = o((\ln t^{-1})^{(1+\epsilon)/2})$. These estimates are sufficient for the analysis of §3 to yield the conclusion of Theorem 4 for $q = 1$. Some of the ideas in the proof of the lemma are motivated by those in [2].

Proof of the lemma. We first develop some relationships which will be needed later in the proof. A and B will denote positive constants, not necessarily the same with each usage. Let $2I = \Sigma m_i \mathbf{r}_i^2$. Then $\ddot{I} = \Sigma m_i \dot{\mathbf{r}}_i^2 + \Sigma m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = 2T + \Sigma \mathbf{r}_i \cdot \partial U / \partial \mathbf{r}_i$. Motivated by Euler's theorem we note that for positive λ,

$U(\lambda \mathbf{r}_1, \cdots, \lambda \mathbf{r}_n) = -\Sigma m_i m_j \ln \lambda + U(\mathbf{r}_1, \cdots, \mathbf{r}_n)$. Differentiating both sides with respect to λ yields $\Sigma \mathbf{r}_i \cdot (\partial U/\partial \mathbf{r}_i)(\lambda \mathbf{r}_1, \cdots, \lambda \mathbf{r}_n) = (-\Sigma^* m_i m_j)/\lambda$. For $\lambda = 1$, $\Sigma \mathbf{r}_i \cdot \partial U/\partial \mathbf{r}_i = -\Sigma^* m_i m_j$. That is,

$$\ddot{I} = 2T - \sum{}^* m_i m_j. \tag{5.2}$$

This is the Lagrange-Jacobi relationship for the inverse force law.

Let $\mathbf{c}_s$ be the center of mass of the particles with limit point $\mathbf{L}_s$. That is, $M_s \mathbf{c}_s = \Sigma_{j \in G_s} m_j \mathbf{r}_j$ where $M_s = \Sigma_{j \in G_s} m_j$ and $s = 1, 2, \cdots, k$. Then

$$M_s \ddot{\mathbf{c}}_s = \sum m_j \ddot{\mathbf{r}}_j = \sum_{j, l \in G_s; j \neq l} \frac{m_l m_j (\mathbf{r}_l - \mathbf{r}_j)}{r_{jl}^2} + \sum_{\alpha=1; \alpha \neq s}^{k} \sum_{l \in G_\alpha} \frac{m_j m_l (\mathbf{r}_l - \mathbf{r}_j)}{r_{jl}^2}.$$

In the first double summation, the scalars are symmetric and the vectors are anti-symmetric with respect to the indices. Hence the terms cancel each other pairwise and the summation is equal to 0. In the second summation each of the $\mathbf{r}_i$'s are approaching distinct limits. That is, $\ddot{\mathbf{c}}_s = O(1)$ as $t \to 0$.

By definition of J, $J = I - \Sigma M_s \mathbf{c}_s \cdot \mathbf{L}_s + \tfrac{1}{2} \Sigma M_s \mathbf{L}_s^2$. That is, $\ddot{J} = \ddot{I} - \Sigma M_s \ddot{\mathbf{c}}_s \cdot \mathbf{L}_s = \ddot{I} + O(1)$. From (5.2) and (4.3) we have

$$\ddot{J} = 2E + O(1) = 2U + O(1). \tag{5.3}$$

We next show that

$$\dot{J}^2 \leq 4JT. \tag{5.4}$$

By definition $|\dot{J}| = |\Sigma \Sigma m_i (\mathbf{r}_i - \mathbf{L}_s) \cdot \dot{\mathbf{r}}_i| \leq \Sigma \Sigma \sqrt{m_i} |\dot{\mathbf{r}}_i| \cdot \sqrt{m_i} |\mathbf{r}_i - \mathbf{L}_s|)$. By the Cauchy inequality $\dot{J}^2 \leq (\Sigma \Sigma m_i (\mathbf{r}_i - \mathbf{L}_s))(\Sigma \Sigma m_i \dot{\mathbf{r}}_i^2)$. (5.4) now follows from the definition of J and T.

$J^{1/2}$ behaves as an upper bound on how fast particles approach each other at collision. To see this, note that if $i, j \in G_s$, $i \neq j$, and if $m_0 = \min(m_i, m_j)$,

$$J \geq m_i (\mathbf{r}_i - \mathbf{L}_s)^2 + m_j (\mathbf{r}_j - \mathbf{L}_s)^2 \geq m_0 ((\mathbf{r}_i - \mathbf{L}_s)^2 + (\mathbf{r}_j - \mathbf{L}_s)^2) \geq m_0 (\mathbf{r}_i - \mathbf{r}_j)^2/2.$$

(The last inequality follows from $2(a^2 + b^2) \geq (a - b)^2$.) That is,

$$r_{ij}^{-1} \geq A J^{-1/2} \quad \text{for } i, j \in G_s. \tag{5.5}$$

Note that if i, j belong to the same G_s, $r_{ij} \to 0$ as $t \to 0$, and if not, then r_{ij} approaches a positive limit. Also, since we are assuming that there is a collision at $t = 0$, some G_s has at least two indices. This implies that $U \to \infty$ as $t \to 0$. Since U (see (5.1)) can be separated uniquely into a sum of the terms where $r_{ij} \to 0$ and a sum of the terms where r_{ij} approaches a positive limit as $t \to 0$, it follows from (5.5) that after some time $U \geq B \ln J^{-1/2}$ where B is some positive constant.

This means that (5.3) can be expressed as $\ddot{J} \geq 2B \ln J^{-1/2} + O(1)$. But $J \to 0$ as $t \to \infty$, so by changing the value of B, after some time

$$\ddot{J} \geq B \ln J^{-1}. \tag{5.6}$$

We now find some of the properties of $\dot{J}$ as $t \to 0$. From (5.6), $\ddot{J} \to \infty$ as $t \to 0$, i.e. J is concave up. So $\dot{J} \to l$ as $t \to 0$ where $-\infty \leq l < \infty$. But $J \to 0$ as $t \to 0$ and by definition $J \geq 0$. Hence, since $\ddot{J} > 0$, $\dot{J}$ is nonnegative in some interval $(0, t)$. Thus $0 \leq l$. We record this as

$$\dot{J} \to l \quad \text{as } t \to 0 \text{ where } l \text{ is a nonnegative constant.} \tag{5.7}$$

Since $\dot{J} \geq 0$, $\dot{J}\ddot{J} \geq B\dot{J} \ln J^{-1}$.

Integrating from t_1 to t_2, $0 < t_1 < t_2$,

$$\dot{J}^2(t_2) - \dot{J}^2(t_1) \geq B(J \ln J^{-1} + J)\Big|_{t_1}^{t_2}.$$

(Recall, B is not necessarily the same positive constant with each usage.)

Let $t_1 \to 0$. Then $\dot{J}^2(t) \geq B(J \ln J^{-1} + J)$ or $\dot{J}/(J \ln J^{-1})^{1/2} \geq B$. That is,

$$J \geq Bt^2 \ln t^{-1}. \tag{5.8}$$

However, (5.8) and the inequality immediately preceding it imply

$$\dot{J} \geq Bt \ln t^{-1}. \tag{5.9}$$

6. **Completion of the proof.** We first show that $\dot{J}/J^{1/2}(\ln J^{-1})^{\beta/2} \to 0$ as $t \to 0$ for some constant $\beta > 1$. By definition of the terms involved and (5.3)

$$\frac{d}{dt}\left(\frac{\dot{J}}{\dot{J}/(J(\ln J^{-1})^{\beta})^{1/2}}\right) = \frac{(J\ddot{J} - \tfrac{1}{2}\dot{J}^2)(\ln J^{-1})^{\beta} + (\beta/2)\dot{J}^2(\ln J^{-1})^{\beta-1}}{(J(\ln J^{-1})^{\beta})^{3/2}}$$

$$= \frac{(4JE - \dot{J}^2)(\ln J^{-1})^{\beta} + \beta\dot{J}^2(\ln J^{-1})^{\beta-1}}{2(J(\ln J^{-1})^{\beta})^{3/2}} + O(1)(J(\ln J^{-1})^{\beta})^{-1/2}.$$

Integrating from t to a, $0 < t < a$,

$$\begin{aligned} &\frac{\dot{J}(a)}{J(\ln J^{-1})^{\beta})^{1/2}} - \frac{\dot{J}(t)}{(J(\ln J^{-1})^{\beta})^{1/2}} \\ &= \int_t^a \frac{(4JE - \dot{J}^2)(\ln J^{-1})^{\beta} + \beta\dot{J}^2(\ln J^{-1})^{\beta-1}\, du}{2(J(\ln J^{-1})^{\beta})^{3/2}} + O(1) \int_t^a \frac{du}{(J(\ln J^{-1})^{\beta})^{1/2}}. \end{aligned} \tag{6.1}$$

By (5.8) the last integral on the right-hand side is

$$\int_t^a (J(\ln J^{-1})^{\beta})^{-1/2}\, du \le B \int_t^a (u(\ln u^{-1})^{\beta})^{-1/2}\, du.$$

That is, for $\beta > 1$ the integral is convergent.

By (5.7), the second term on the left-hand side of (6.1) is positive, so by dropping it we have an inequality. Since the error term converges, it is bounded and we have from (6.1) that the first integral on the right-hand side is bounded above. But by (5.4), the integrand of this integral is nonnegative. Since the integral is bounded above, it converges.

The above shows that the right-hand side of (6.1) converges as $t \to 0$, hence the left-hand side does. That is, $\dot{J}/(J(\ln J^{-1})^{\beta})^{1/2}$ has a limit as $t \to 0$ for $\beta > 1$. Since this is true for all $\beta > 1$ and since $J \to 0$ as $t \to 0$ (i.e. $(\ln J^{-1})^{\beta} \to \infty$), it follows that $\dot{J}/(J(\ln J^{-1})^{\beta})^{1/2} \to 0$ as $t \to 0$.

By integrating this relationship, we have that

$$J = o(t^2(\ln t^{-1})^{\beta}) \quad \text{as } t \to 0 \text{ for some } \beta > 1. \tag{6.2}$$

This is the first part of the conclusion of the lemma. It remains to obtain the estimates on T as $t \to 0$. This is obtained from the above by a Tauberian type argument.

Note that from the above derived relationship and (6.2),

$$\dot{J} = o(J^{1/2}(\ln J^{-1})^{\beta/2}) = o(t(\ln t^{-1})^{\beta}) \quad \text{for } \beta > 1. \tag{6.3}$$

To show that the estimates on E hold, it follows from (5.8) that it is sufficent to show for $\epsilon > 0$ that $\ddot{J}/(\ln J^{-1})^{1+2\epsilon} \to 0$ as $t \to 0$. But

$$\ddot{J}/(\ln J^{-1})^{1+2\epsilon} = (t\ddot{J}/\dot{J}(\ln J^{-1})^{\epsilon})(\dot{J}/t(\ln J^{-1})^{1+\epsilon}). \tag{6.4}$$

By (6.2) and (5.8), there exist positive constants A and B such that after some time $A \ln t^{-1} \le \ln J^{-1} < B \ln t^{-1}$. Hence from (6.3) it follows that the second term on the right-hand side of (6.4) approaches zero.

So, in order to prove the statement, it suffices to show that $\limsup(t\ddot{J}/\dot{J}(\ln J^{-1})^{\epsilon}) < \infty$ as $t \to 0$. By our knowledge of the behavior of $\ln J^{-1}$, this is true if $\limsup(t\ddot{J}/\dot{J}(\ln t^{-1})^{\epsilon}) < \infty$. Assume this to be false. Then there exist intervals $[t_1, t_2]$ arbitrarily close to zero such that for $t \in [t_1, t_2]$, $\ddot{J}/\dot{J} \ge (\ln t^{-1})^{\epsilon}/t = -(\ln t^{-1})^{\epsilon} d(\ln t^{-1})$, or $\ln \dot{J}(t_2) - \ln \dot{J}(t_1) \ge (\ln t_1^{-1})^{\epsilon+1} - (\ln t_2^{-1})^{1+\epsilon}$. But (5.9) and (6.3) imply that $\ln \dot{J}(t) = \ln t(1 + o(1))$. That is,

$$\ln t_1^{-1}(1 + o(1)) - \ln t_2^{-1}(1 + o(1)) \ge (\ln t_1^{-1})^{1+\epsilon} - (\ln t_2^{-1})^{1+\epsilon}.$$

This is clearly false for t_1, t_2 close to zero. The proof is completed.

7. **Binary collisions.** In this section we develop the analytic behavior of binary collisions in the inverse q force law and indicate the changes in §§2, 3 necessary to prove Theorem 3. Since Theorem 4 has already been proved, for $q \leq 1$, we restrict our attention to $1 < q < 3$. In this setting it is known [3, p. 239] that if $\mathbf{r}_1$ and $\mathbf{r}_2$ collide (in a binary or higher order collision) at time t^* then there exist positive constants A, B and C such that after some time

$$(7.1)\quad A|t - t^*|^{2/(q+1)} \leq |\mathbf{r}_1 - \mathbf{r}_2| \leq B|t - t^*|^{2/(q+1)} \text{ and } |\dot{\mathbf{r}}_1|, |\dot{\mathbf{r}}_2| \leq C|t - t^*|^{(1-q)/(q+1)}.$$

Assume that $\mathbf{r}_1$ and $\mathbf{r}_2$ collide in a binary collision at collision point $\mathbf{L}$ at collision time t^*. We then define $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. From (4.1), it follows that the equations of motion for $\mathbf{r}$ are

$$(7.2)\qquad \ddot{\mathbf{r}} = -\mu\mathbf{r}/r^{q+1} + O(1) \quad \text{as } t \to t^*$$

where μ is a positive constant depending on the masses. This is so since the magnitude of each term on the right-hand side of (4.1) is of the order r_{ij}^q. Since we have a binary collision, the distance between $\mathbf{r}_1$ or $\mathbf{r}_2$ and $\mathbf{r}_j$ where $j \neq 1, 2$, is bounded away from zero. From this, (7.2) follows immediately. (Note, if $\mathbf{r}_j$ collides with $\mathbf{r}_k$, then $k \neq 1, 2$.)

From (7.1) and (7.2) an angular momentum relationship can be derived. To accomplish this, note that $\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + O(|\mathbf{r}|) = O(|t - t^*|^{2/(q+1)})$. Integrating from t_1 to t_2, $t_1 < t_2 < t^*$, and defining $\mathbf{c} = \mathbf{r} \times \mathbf{v}$, it follows that $\mathbf{c}(t_2) - \mathbf{c}(t_1) = O(|t - t^*|^{(3+q)/(q+1)})$. As $t_1, t_2 \to 0$, the right-hand side goes to zero, carrying the left-hand side with it. By the Cauchy criterion for the existence of a limit, $\mathbf{c}(t)$ has a limit as $t \to t^*$. But since $|\mathbf{c}| \leq |\mathbf{r}|\,|\mathbf{v}|$, it follows from (7.1) that $\mathbf{c} \to \mathbf{0}$ as $t \to t^*$. That is,

$$(7.3)\qquad \mathbf{c}(t) = O(|t - t^*|^{(3+q)/(q+1)}) \quad \text{as } t \to t^*.$$

A first estimate on the growth of $\mathbf{r}$ can now be found. By definition of the terms (7.1) and (7.3)

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}r - \dot{r}\mathbf{r}}{r^2} = \frac{r^2\dot{\mathbf{r}} - r\dot{r}\mathbf{r}}{r^3} = \frac{(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}}{r^3} = \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{r^3}$$

$$= \frac{\mathbf{c}\times\mathbf{r}}{r^3} = \frac{1}{r^2}\,O(|t - t^*|^{(3+q)/(q+1)}) = O(|t - t^*|^{(q-1)/(q+1)}).$$

By integrating and using the Cauchy criterion for the existence of a limit, $\mathbf{r}/r = \mathbf{b} + O(|t - t^*|^{2q/(q+1)})$ or

$$(7.4)\qquad \mathbf{r} = \mathbf{b}r + O(|t - t^*|^2) \text{ where } |\mathbf{b}| = 1.$$

We now obtain a refined estimate on the behavior of r. Recall that for vector $\mathbf{u}$, $\ddot{u} = \ddot{\mathbf{u}} \cdot (\mathbf{u}/u) + (\mathbf{u} \times \dot{\mathbf{u}})^2/u^3$, so from (7.1), (7.2) and (7.3) $\ddot{r} = -\mu/r^q + O(1)$. As $t \to t^*$ it follows that $\ddot{r} \to -\infty$, $r > 0$ and $r \to 0$. Thus, after some time, $\dot{r}$ becomes and remains negative. We then obtain $\dot{r}\ddot{r} = -\mu\dot{r}/r^q + O(|\dot{r}|)$. Integrating from a to t, $a < t < t^*$, we obtain $\dot{r}^2 = Ar^{1-q} + O(1)$ where A is a positive constant. Hence $|r^{(q-1)/2}\dot{r}| = A + O(r^{q-1}) = A + O(|t - t^*|^{2(q-1)/(q+1)})$. Integrating again, we find that

$$r^{(q+1)/2} = A|t - t^*|(1 + O(|t - t^*|^{2(q-1)/(q+1)}))$$

or

$$r = A|t - t^*|^{2/(q+1)}(1 + O(|t - t^*|^{2(q-1)/(q+1)}))$$

where A is a positive constant (obviously not the same as before but a function of the masses and q). Combined with (7.4) this gives

$$\mathbf{r} = A\mathbf{b}|t - t^*|^{2/(q+1)} + O(|t - t^*|^{2q/(q+1)}). \tag{7.5}$$

Substituting this value into (7.2), we see that $\ddot{\mathbf{r}} = -C\mathbf{b}|t - t^*|^{-2q/(q+1)} + O(|t - t^*|^{-2/(q+1)})$. Integrating from t_1 to t_2, $t_1 < t_2 < t^*$, shows that

$$\left(\dot{\mathbf{r}}(t) - \frac{q+1}{q-1}C\mathbf{b}|t - t^*|^{(1-q)/(q+1)}\right)\Bigg|_{t_1}^{t_2} = O(|t_1 - t^*|^{(q-1)/(q+1)}).$$

Since $q > 1$, t_1, $t_2 \to t^*$ implies that the right-hand side approaches zero. This forces the left-hand side to zero. By the Cauchy criterion for the existence of a limit this implies

$$\dot{\mathbf{r}} = \frac{q+1}{q-1}\,C\mathbf{b}|t - t^*|^{(1-q)/(1+q)} + \mathbf{d} + O(|t - t^*|^{(q-1)/(q+1)}). \tag{7.6}$$

Ingegrating yields

$$\mathbf{r} = ((q+1)^2/2(q-1))C\mathbf{b}|t - t^*|^{2/(q+1)} + \mathbf{d}|t - t^*| + O(|t - t^*|^{2q/(q+1)}).$$

Comparing this with (7.5) yields

$$\dot{\mathbf{r}} = (2/(q+1))A\mathbf{b}|t - t^*|^{(1-q)/(1+q)} + O(|t - t^*|^{(q-1)/(q+1)}), \tag{7.7}$$

where A is the same as in (7.5).

From the definition of $\mathbf{r}$ it follows that $\mathbf{r}_1 = -D_1\mathbf{r} + \mathbf{L}$ and $\mathbf{r}_2 = D_2\mathbf{r} + \mathbf{L}$ where D_1, D_2 are positive constants depending on m_1, m_2. We will call $-D_1A\mathbf{b} = \mathbf{c}_1$ and $D_2A\mathbf{b} = \mathbf{c}_2$.

We now can use these estimates to prove Theorem 3. Note first that

$$(0, 0, \ldots, 0, c_1, c_2, 0, \ldots, 0)2(2^{-\alpha})^{(1-q)/(1+q)}/(1+q)$$

defines a smooth two dimensional manifold in the 6 dimensional subspace of the velocity vectors $(0, 0, \cdots, 0, v_1, v_2, 0, \cdots, 0)$. Call this manifold $\mathfrak{M}(\alpha)$. Then following the ideas and notation (with obvious modifications) of §§2 and 3, redefine conditions 3 and 4 of $B^{\alpha}(t)$ to be

(3) $|r_i - L| \leq (|c_1| + |c_2|)(2^{-\alpha})^{2/(q+1)}$ for $L \in L$, and

(4) $(0, \cdots, 0, v_1, v_2, 0, \cdots, 0)$ is within distance $(2^{-\alpha})^{(q-5/4)/(q+1)}$ of $\mathfrak{M}(\alpha)$.

The measure of $B^{\alpha}(t)$ turns out to be less than $E(2^{-\alpha})^{(3+2q)/(q+1)}$ where E is some positive constant.

The time interval $[1, 2]$ is then divided into 2^{β} equal parts where β is a positive integer such that $\beta > \alpha$ and $(3+2q)/(q+1) > \beta/\alpha > (3q - 5/4)/(q+1)$. For large α and $1 < q < 3$, this always can be done.

It then follows that $mB^{\alpha}[1, 2] \leq E(2^{-\alpha})^{((3+2q)/(q+1))-(\beta/\alpha)}$. The lower bound on β/α is needed to verify that the velocity condition is satisfied for some partition point in $[1, 2]$ for all α after some integer. The remainder of the proof follows directly.

The proof of Corollary 2 follows from the above and (7.1).

8. **Higher order collisions.** What remains to be done in order to prove the conjecture for $q < 3$ is to develop the analytic behavior of higher order collisions to approximately the same degree of precision as in (7.5) and (7.7). As pointed out in [3, p. 239], the inverse q force law for $1 < q < 3$ has the behavior near collisions as given by (7.1).

Let $r_1, r_2, \cdots, r_k$ collide at a common collision point L at time t^*. Define $R_i = (r_i - L)/(t^* - t)^{2/(q+1)}$. By mimicking the argument of [3, pp. 229–231] where $t \to t^*$ instead of $t \to \infty$, it follows that

$$\frac{2(1-q)}{(q+1)^2} R_i = \sum_{j=1; j\neq i}^{k} \frac{M_j(R_j - R_i)}{R_{ij}^{q+1}} + o(1) \quad \text{for } i = 1, 2, \cdots, k.$$

That is, as the particles collide, they tend toward a (shrinking) *central configuration.* (Compare this with (1.3).) From this we can obtain that $|r_i - L| = D_i|t - t^*|^{2/(q+1)}(1 + o(1))$ where D_i, $i = 1, 2, \cdots, k$, are nonnegative constants, at most one is zero.

To obtain an improvement on this estimate, which is necessary to employ the present ideas for the proof of the conjecture, it is necessary to find the central configurations and their properties. To my knowledge even for $q = 2$ and $k > 4$ *this information is missing*! Even such an elementary question as *whether there is a finite number of solutions for given masses is unknown.*

The answers to these questions about central configurations would have consequences in other settings of the n-body problem. In [3] this information could be used in the discussion of "subsystems" for the general question of how Newton's universe evolves at $t \to \infty$. In [7] this is needed to show whether there is a finite number of bifurcation sets.

Added in proof. It can also be shown that the set of initial conditions leading to collision is of (Baire) first category.

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